ECED 3300
Tutorial 9

Problem 1

Determine the magnetic field at the center O of a circular loop of radius R with a cut out segment as shown in the figure below. The loop carries the current I.

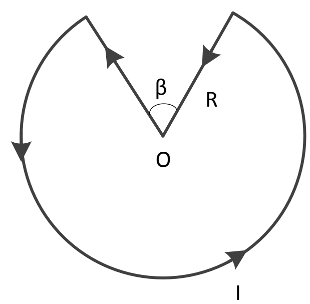

FIG. 1: Illustration to Problem 1.

Solution

Applying Bio-Savart's law,

$$\mathbf{H} = \frac{I}{4\pi} \oint \frac{d\mathbf{l} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}.$$

By the superposition, there're contributions due to the radial paths and the circumference path. We consider them separately.

- Radial paths: $d\mathbf{l} = \mathbf{a}_\rho d\rho$; $\mathbf{r} - \mathbf{r}' = \rho \mathbf{a}_\rho$. It follows that $d\mathbf{l} \times (\mathbf{r} - \mathbf{r}') = \rho d\rho (\mathbf{a}_\rho \times \mathbf{a}_\rho) = 0$.

- Circumference path: $d\mathbf{l} = \mathbf{a}_\phi R d\phi$; $\mathbf{r} - \mathbf{r}' = -R\mathbf{a}_\rho$. Thus, $d\mathbf{l} \times (\mathbf{r} - \mathbf{r}') = -R^2 d\phi (\mathbf{a}_\phi \times \mathbf{a}_\rho) = R^2 d\phi \mathbf{a}_z$.

Therefore,

$$\mathbf{H} = \frac{I}{4\pi} \int_0^{2\pi - \beta} \frac{R^2 d\phi \mathbf{a}_z}{R^3} = \frac{I}{4\pi R}(2\pi - \beta)\mathbf{a}_z.$$

Problem 2

Given the vector field $\mathbf{F}$ *such that*

$$\mathbf{F} = \frac{-y\mathbf{a}_x + x\mathbf{a}_y}{x^2 + y^2}.$$

a) Determine whether $\mathbf{F}$ *can represent a magnetic flux density.*

b) Determine the flux of $\mathbf{F}$ *through a cylinder of height* H *and radius* R*, centered at the origin.*

Solution

a) The magnetic flux density must be solenoidal, $\nabla \cdot \mathbf{B} = 0$ (no magnetic charges). A simple check confirms that $\nabla \cdot \mathbf{F} = \partial_x F_x + \partial_y F_y = 2xy/(x^2+y^2)^2 - 2yx/(x^2+y^2)^2 = 0$.

b) The flux is equal to zero because the field is solenoidal; in other words, using Gauss's theorem

$$\oint d\mathbf{S} \cdot \mathbf{F} = \int dv \nabla \cdot \mathbf{F} = 0.$$

Alternatively, converting to the cylindrical coordinates yields, $\mathbf{F} = \mathbf{a}_\phi/\rho$. Since $\mathbf{a}_\phi$ is orthogonal to the outward unit normals to the top/bottom, $\pm\mathbf{a}_z$, and the wall, $\mathbf{a}_\rho$ surfaces of the cylinder, the overall flux vanishes.

Problem 3

An infinitely long cylinder of radius a and permeability μ is placed such that its axis coincides with the z-axis. The cylinder carries a current I uniformly distributed across. The current is in the z-direction as well. Find

(i) magnetization inside the cylinder,

(ii) volume magnetization current density inside the cylinder.

Solution

Guessing the direction of magnetic field to be azimuthal from the symmetry of the problem and applying Ampère's law in the integral form, $\oint_L \mathbf{H} \cdot d\mathbf{l} = I_{enc}$, to any circular Ampèrian path of radius $\rho \leq a$, we obtain

$$H \times 2\pi\rho = \underbrace{\frac{I}{\pi a^2}}_{current\ density} \times \pi\rho^2 \Longrightarrow \mathbf{H} = \frac{I\rho}{2\pi a^2}\mathbf{a}_\phi.$$

It then follows that $\mathbf{B} = \mu\mathbf{H} = (\mu I\rho/2\pi a^2)\mathbf{a}_\phi$. The magnetization is determined as

$$\mathbf{M} = \frac{\mathbf{B}}{\mu} - \mathbf{H} = \left(\frac{\mu}{\mu_0} - 1\right)\frac{I\rho}{2\pi a^2}\mathbf{a}_\phi.$$

The magnetization current density can be found from its definition,

$$\mathbf{J}_m = \nabla \times \mathbf{M} = \frac{1}{\rho}\begin{vmatrix} \mathbf{a}_\rho & \rho\mathbf{a}_\phi & \mathbf{a}_z \\ \partial_\rho & \partial_\phi & \partial_z \\ 0 & \frac{I\rho^2(\mu/\mu_0 - 1)}{2\pi a^2} & 0 \end{vmatrix}.$$

Thus, calculating the curl, we arrive at

$$\mathbf{J}_m = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\frac{I\rho^2(\mu/\mu_0 - 1)}{2\pi a^2}\right]\mathbf{a}_z = \frac{I}{\pi a^2}\left(\frac{\mu}{\mu_0} - 1\right)\mathbf{a}_z.$$